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LETTER TO THE EDITOR

Hidden integrability of a quantum system with non-local coupling

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Abstract. A system of polaritons interacting with a two-level atom placed within a frequency dispersive medium is proved to be integrable, despite a non-local effective polariton–polariton coupling. The two-polariton factorization of a many-polariton scattering process is hidden and is manifested only in the limit of large interpolariton separations.

The standard Dicke and Bloch–Maxwell models [1], which describe a system of photons coupled to two-level atoms, are integrable and can be diagonalized exactly [2–4] by means of the Bethe-ansatz technique [5, 6]. In the present letter, we study a quantum system of polaritons ('photons in a medium') [7] interacting with a single two-level atom placed within a frequency dispersive medium. The polariton–atom coupling is *non-local* and leads to a *non-local* effective polariton–polariton coupling. Therefore, the integrability of the 'polaritons + atom' system is highly questionable and requires a thorough analysis.

To diagonalize the model Hamiltonian, we introduce *auxiliary* particles and show that a many-particle scattering process is factorized into two-particle ones. The two-polariton factorization of many-polariton scattering is *hidden* and is manifested only in the limit of large interpolariton separations.

In the dipole resonance (rotating wave) approximation [1] the model Hamiltonian is written as

$$
\begin{aligned}
H = \omega_{12}\sum_{\sigma} X_{\sigma\sigma} + \sum_{\alpha\sigma}\int_0^\infty \frac{\mathrm{d}k}{2\pi}\,\epsilon_\alpha(k)\,p^+_{\alpha\sigma}(k)\,p_{\alpha\sigma}(k) \\
- \sum_{\alpha\sigma}\int_0^\infty \frac{\mathrm{d}k}{2\pi}\sqrt{\gamma(k)}\left(\frac{|\Omega-\epsilon_\alpha(k)|}{\epsilon_2(k)-\epsilon_1(k)}\right)^{1/2}\left[p_{\alpha\sigma}(k)X_{\sigma 0} + X_{0\sigma}\,p^+_{\alpha\sigma}(k)\right]. \qquad (1)
\end{aligned}
$$

Here the index $\alpha = 1, 2$ enumerates the polariton branches of the frequency dispersive medium with the spectral functions

$$
\epsilon_1(k) = \tfrac{1}{2}\left[(\Omega+k) - \sqrt{(\Omega-k)^2 + 4k\Delta}\right] \qquad \epsilon_2(k) = \tfrac{1}{2}\left[(\Omega+k) + \sqrt{(\Omega-k)^2 + 4k\Delta}\right]
$$

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and $\gamma(k) = 4k^3d^2/3$, where ω_{12} and d are respectively the frequency and the dipole moment of the atomic transition. The operators $p_{\alpha\sigma}(k)$ with the commutation relations

$$[p_{\alpha\sigma}(k), p^+_{\alpha'\sigma'}(k')] = 2\pi\delta_{\alpha\alpha'}\delta_{\sigma\sigma'}\delta(k-k')$$

describe the electro-dipole harmonics [8] of the entire polariton field. The 'colour' index $\sigma = 1, 2, 3$ enumerates both the three degenerate states of the excited atomic level and the electro-dipole harmonics of the polariton field with three possible projections of the angular momentum $m = -1, 0, 1$ [4]. The atom is described by the Hubbard operators X_{ab}, $a, b = (0, \sigma)$ with the commutator

$$[X_{ab}, X_{cd}] = \delta_{cb}X_{ad} - \delta_{ad}X_{cb}$$

where the index 0 stands for the atomic ground state. The polariton frequency varies from zero to $\Omega - \Delta$ within the lower branch and from Ω to $+\infty$ within the upper branch. The frequency interval of the width Δ between $\Omega - \Delta$ and Ω is forbidden for propagating polariton modes.

Introducing the energy variable ϵ by the expression $k = \epsilon(\Omega - \epsilon)/(\Omega - \Delta - \epsilon)$ and the polariton operators on the 'energy scale'

$$p_\sigma(\epsilon_\alpha(k)) = \left(\frac{\epsilon_2(k) - \epsilon_1(k)}{|\Omega - \Delta - \epsilon_a(k)|}\right)^{1/2} p_{\alpha\sigma}(k) \qquad [p_\sigma(\epsilon), p^+_{\sigma'}(\epsilon')] = 2\pi\delta_{\sigma\sigma'}\delta(\epsilon - \epsilon')$$

one can turn from integration WRT k in (1) to integration WRT ϵ to obtain

$$H = \omega_{12}\sum_\sigma X_{\sigma\sigma} + \sum_\sigma \int_C \frac{d\epsilon}{2\pi}\left\{\epsilon\, p^+_\sigma(\epsilon)p_\sigma(\epsilon) - \sqrt{\gamma}\, z(\epsilon)\,[p_\sigma(\epsilon)X_{\sigma 0} + X_{0\sigma}p^+_\sigma(\epsilon)]\right\}. \qquad (2)$$

The atomic form factor $z(\epsilon) = (\Omega - \epsilon)^2/[(\Omega - \Delta - \epsilon)^2 + \kappa^2]$ reflects the growth of $\gamma(k)$ and the density of polariton states near the upper edge of the lower polariton branch. The constant κ is introduced to account for relaxation processes in the medium. In accordance with the resonance approximation, here we have replaced $\gamma(\epsilon) \to \gamma = 4\omega_{12}^3 d^2/3 = \text{constant}$ and extended the lower limit of integration to $-\infty$. Thus, the integration contour in (2) consists of two semi-infinite intervals, $C = (-\infty, \Omega - \Delta] \bigcup [\Omega, \infty)$.

The one-particle eigenstates of the model

$$|\lambda\rangle = \sum_\sigma \beta_\sigma \left[\mathrm{g}(\lambda)X_{\sigma 0} + \int_C \frac{d\epsilon}{2\pi} f(\epsilon|\lambda)p^+_\sigma(\epsilon)\right]|0\rangle \qquad (3)$$

where the β_σ are arbitrary constants and the vacuum state is defined by

$$X_{0\sigma}|0\rangle = p_\sigma(\epsilon)|0\rangle = 0$$

are found from the Schrödinger equation

$$(\epsilon - \lambda)f(\epsilon|\lambda) - \sqrt{\gamma}z(\epsilon)\mathrm{g}(\lambda) = 0 \qquad (\lambda - \omega_{12})\mathrm{g}(\lambda) + \sqrt{\gamma}\int_C \frac{d\epsilon}{2\pi} z(\epsilon)f(\epsilon|\lambda) = 0. \qquad (4)$$

Their spectrum consists of both the continuous spectrum with eigenenergy λ lying outside the gap:

$$f(\epsilon, \lambda) = 2\pi z(\lambda)\delta(\epsilon - \lambda) + \sqrt{\gamma}\frac{z(\epsilon)}{\epsilon - \lambda - \mathrm{i}0}\mathrm{g}(\lambda) \qquad (5a)$$

$$\mathrm{g}(\lambda) = -\frac{\sqrt{\gamma}z^2(\lambda)}{\lambda - \omega_{12} + \Sigma(\lambda)} \qquad \Sigma(\lambda) = \gamma\int_C \frac{d\epsilon}{2\pi}\frac{z^2(\epsilon)}{\epsilon - \lambda - \mathrm{i}0} \qquad (5b)$$

and the discrete mode:

$$f_d(\epsilon|\Lambda) = \sqrt{\gamma}\,\frac{z(\epsilon)}{\epsilon - \Lambda}\,\mathrm{g}_d(\Lambda) \qquad \Sigma_d(\Lambda) = \gamma \int_C \frac{\mathrm{d}\epsilon}{2\pi}\,\frac{z^2(\epsilon)}{\epsilon - \Lambda} \qquad \mathrm{Im}\,\Sigma_d(\Lambda) = 0. \tag{6a}$$

The eigenenergy of the discrete state Λ lies within the gap and is found as a root of the equation

$$\Lambda - \omega_{12} + \Sigma(\Lambda) = 0 \qquad \Lambda \in [\Omega - \Delta, \Omega]. \tag{6b}$$

The modulus of arbitrary value $\mathrm{g}_d(\Lambda)$ is determined from the normalization condition $\langle\Lambda|\Lambda\rangle = 1$.

For what follows, it is convenient to rewrite equation (3) in terms of the Fourier transform of the polariton operators and wavefunctions:

$$p_\sigma^+(x) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\epsilon}{2\pi}\,p_\sigma^+(\epsilon)\mathrm{e}^{-\mathrm{i}\epsilon x} \qquad f(x|\lambda) = \int_{-\infty}^{\infty} \frac{\mathrm{d}\epsilon}{2\pi}\,f(\epsilon|\lambda)\mathrm{e}^{\mathrm{i}\epsilon x}\,.$$

We then obtain

$$|\lambda\rangle = \sum_\sigma \beta_\sigma \left[\mathrm{g}(\lambda)X_{\sigma 0} + \int_{-\infty}^{\infty} \mathrm{d}x\ \psi(x|\lambda)p_\sigma^+(x)\right]|0\rangle \qquad \psi(x|\lambda) = \int_C \frac{\mathrm{d}\epsilon}{2\pi}\,f(\epsilon|\lambda)\mathrm{e}^{\mathrm{i}\epsilon x}\,. \tag{7}$$

Note that the polariton wavefunction in the auxiliary x-space $\psi(x)$ is not equal to the function $f(x)$, due to the existence of the gap. Therefore we introduce the auxiliary function $\phi(\epsilon|\lambda) = z^{-1}(\epsilon)f(\epsilon|\lambda)$, and represent $\psi(x)$ as

$$\psi(x|\lambda) = \int_{-\infty}^{\infty} \mathrm{d}y\ u(x-y)\phi(y|\lambda) \qquad u(x) = \int_C \frac{\mathrm{d}\epsilon}{2\pi}\,z(\epsilon)\mathrm{e}^{\mathrm{i}\epsilon x}\,. \tag{8}$$

In the auxiliary space, the Schrödinger equation for the wavefunction of the auxiliary particle takes the form

$$(-\mathrm{i}\partial_x - \lambda)\,\phi(x|\lambda) = \sqrt{\gamma}\mathrm{g}(\lambda)\delta(x). \tag{9}$$

Equation (9) describes the auxiliary particle propagating in the positive direction of the x-axis and scattering on the point-like potential. Its general solution is given by

$$\phi(x|\lambda) = \frac{h(\lambda) - (\mathrm{i}/2)\mathrm{sgn}(x)}{h(\lambda) + \mathrm{i}/2}\,\mathrm{e}^{\mathrm{i}\lambda x} \qquad h(\lambda) = \frac{\lambda - \omega_{12} + \Sigma'(\lambda)}{\gamma z^2(\lambda)} \tag{10}$$

where $\mathrm{sgn}(x) = (-1, x < 0; 0, x = 0; 1, x > 0)$ and $\Sigma'(\lambda) = \mathrm{Re}\,\Sigma(\lambda)$.

We look for N-particle eigenstates in the form

$$|\Psi_N\rangle = \sum_{\sigma_1,\ldots\sigma_N} \left[\int_{-\infty}^{\infty} \mathrm{d}x_1\cdots\mathrm{d}x_N\ \Psi_{\sigma_1,\ldots,\sigma_N}(x_1,\ldots,x_N)\prod_{j=1}^{N} p_{\sigma_j}^+(x_j)\right.$$
$$\left. + \int_{-\infty}^{\infty} \mathrm{d}x_1\cdots\mathrm{d}x_{N-1}\ J_{\sigma_1,\ldots,\sigma_N}(x_1\ldots,x_{N-1})X_{\sigma_N 0}\prod_{j=1}^{N-1} p_{\sigma_j}^+(x_j)\right]|0\rangle \tag{11}$$

where the polariton wavefunctions are also expressed in terms of the wavefunctions of the auxiliary particles:

$$\Psi_{\sigma_1,\ldots,\sigma_N}(x_1,\ldots,x_N) = \int_{-\infty}^{\infty} \mathrm{d}y_1\cdots\mathrm{d}y_N\ \Phi_{\sigma_1,\ldots,\sigma_N}(y_1,\ldots,y_N)\prod_{j=1}^{N} u(x_j - y_j) \tag{12a}$$

$$J_{\sigma_1,\ldots,\sigma_N}(x_1,\ldots,x_{N-1}) = \int_{-\infty}^{\infty} \mathrm{d}y_1\cdots\mathrm{d}y_{N-1}\ G_{\sigma_1,\ldots,\sigma_N}(y_1,\ldots,y_{N-1})\prod_{j=1}^{N-1} u(x_j - y_j)\,. \tag{12b}$$

In the two-particle case, the auxiliary functions obey the Schrödinger equation

$$(-\mathrm{i}\partial_{x_1} - \mathrm{i}\partial_{x_2} - E)[\Phi_{\sigma_1\sigma_2}(x_1, x_2) + \Phi_{\sigma_2\sigma_1}(x_2, x_1)]$$
$$= \sqrt{\gamma}\,[\delta(x_1)G_{\sigma_1\sigma_2}(x_2) + G_{\sigma_2\sigma_1}(x_1)\delta(x_2)] \tag{13a}$$

$$(-\mathrm{i}\partial_x + \omega_{12} - E)G_{\sigma_1\sigma_2}(x) = \sqrt{\gamma}\int_{-\infty}^{\infty}\mathrm{d}x'\ v(x')[\Phi_{\sigma_1\sigma_2}(x, x') + \Phi_{\sigma_2\sigma_1}(x', x)] \tag{13b}$$

where

$$v(x) = \int_C \frac{\mathrm{d}\epsilon}{2\pi}\ z^2(\epsilon)\,\mathrm{e}^{-\mathrm{i}\epsilon x}\,.$$

We look for a solution of (13) in the form

$$\Phi_{\sigma_1\sigma_2}(x_1, x_2|\lambda_1, \lambda_2) = \sum_{\sigma_1'\sigma_2'} A_{\sigma_1\sigma_2}^{\sigma_1'\sigma_2'}(x_1, x_2)\phi_{\sigma_1'}(x_1|\lambda_1)\phi_{\sigma_2'}(x_2|\lambda_2) \tag{14a}$$

$$G_{\sigma_1\sigma_2}(x|\lambda_1, \lambda_2) = \sum_{\sigma_1'\sigma_2'}\Big[A_{\sigma_1\sigma_2}^{\sigma_1'\sigma_2'}(x)\phi_{\sigma_1'}(x|\lambda_1)\mathrm{g}_{\sigma_2'}(\lambda_2) + A_{\sigma_1\sigma_2}^{\sigma_1'\sigma_2'}(-x)\mathrm{g}_{\sigma_1'}(\lambda_1)\phi_{\sigma_2'}(x|\lambda_2)\Big] \tag{14b}$$

where $E = \lambda_1 + \lambda_2$.

In the unicolour case ($\sigma_1 = \sigma_2$), the function $A(x)$ is found to be

$$A(x_1, x_2|\lambda_1, \lambda_2) = 1 + \frac{\mathrm{i}}{h(\lambda_1) - h(\lambda_2)}\,\mathrm{sgn}(x_1 - x_2)\,.$$

Thus the particle–particle scattering is described by the discontinuous jump at the permutation of particle coordinates, while the corresponding polariton wavefunctions (12) are continuous. In the general colour case, the two-particle scattering matrix is found to be

$$S_{\sigma_1\sigma_2}^{\sigma_1'\sigma_2'}(\lambda_1, \lambda_2) = a(\lambda_1, \lambda_2) + b(\lambda_1, \lambda_2)P_{\sigma_1\sigma_2}^{\sigma_1'\sigma_2'} \tag{15a}$$

where

$$a(\lambda_1, \lambda_2) = \frac{h(\lambda_1) - h(\lambda_2)}{h(\lambda_1) - h(\lambda_2) - \mathrm{i}\gamma} \qquad b(\lambda_1, \lambda_2) = \frac{\mathrm{i}\gamma}{h(\lambda_1) - h(\lambda_2) - \mathrm{i}\gamma}\,. \tag{15b}$$

and $P_{\sigma_1\sigma_2}^{\sigma_1'\sigma_2'} = \delta_{\sigma_1\sigma_2'}\delta_{\sigma_2\sigma_1'}$ is the permutation operator. The two-particle scattering matrix is obviously a solution of the Yang–Baxter equations [5, 6, 9], and hence, the many-particle scattering is factorized into two-particle ones. The two-polariton factorization of the many-polariton scattering is hidden due to non-local coupling in the polariton system, and becomes visible only in the limit of large interpolariton separations.

To find the spectrum of the system, we have to put the system in a 'box' of size L and to impose the periodic boundary conditions (PBC) on the polariton wavefunction. The PBC lead to a hierarchy of the Bethe-ansatz equations, which can be obtained from the hierarchy of the Bethe-ansatz equations in the colour Dicke model [4] by introducing of the 'rapidities' $h_j \equiv h(\lambda_j)$ (10). For the sake of simplicity, we confine ourselves here to the unicolour case, where all the polaritons have the same colour. Then, the Bethe-ansatz equations are given by

$$\mathrm{e}^{\mathrm{i}k(\lambda_j)L}\,\frac{h(\lambda_j) - \mathrm{i}/2}{h(\lambda_j) + \mathrm{i}/2} = -\prod_{l=1}^{N}\frac{h(\lambda_j) - h(\lambda_l) - \mathrm{i}}{h(\lambda_j) - h(\lambda_l) + \mathrm{i}} \qquad E = \sum_{j=1}^{N}\lambda_j\,. \tag{16a}$$

Here, the polariton wavevector $k(\lambda) = \lambda(\Omega-\lambda)/(\Omega-\Delta-\lambda)$ describes the spatial behaviour of wavefunctions. If one of the polaritons is bound to the atom, equations (16a) take the form

$$\mathrm{e}^{\mathrm{i}k(\lambda_j)L}\,\frac{h(\lambda_j)-\mathrm{i}/2}{h(\lambda_j)+\mathrm{i}/2}\,\frac{h(\lambda_j)+\mathrm{i}}{h(\lambda_j)-\mathrm{i}} = -\prod_{l=1}^{N-1}\frac{h(\lambda_j)-h(\lambda_l)-\mathrm{i}}{h(\lambda_j)-h(\lambda_l)+\mathrm{i}} \qquad E = \Lambda + \sum_{j=1}^{N-1}\lambda_j\,. \tag{16b}$$

In the limit $L \to \infty$, apart from real solutions, equations (16) admit complex ones, in which rapidities h_j are grouped into 'strings':

$$h_j^{(\alpha,n)} = h^{(\alpha,n)} + \tfrac{1}{2}\mathrm{i}(n+1-2j) \qquad j = 1,\ldots,n \tag{17}$$

where $h^{(\alpha,n)}$ is a common real part, and n is the order of a string. The relationship $h(\lambda)$ is obtained by the analytical continuation of (10) in the complex λ plane:

$$h(\lambda) = \begin{cases} [\gamma z^2(\lambda)]^{-1}[\lambda - \omega_{12} + \Sigma(\lambda) - (\mathrm{i}\gamma/2)z^2(\lambda)] & \mathrm{Im}\,\lambda > 0 \\ [\gamma z^2(\lambda)]^{-1}[\lambda - \omega_{12} + \Sigma(\lambda) + (\mathrm{i}\gamma/2)z^2(\lambda)] & \mathrm{Im}\,\lambda < 0\,. \end{cases} \tag{18}$$

For h_j lying far from the real axis, one gets $h_j \sim (\lambda_j - \omega_{12})/\gamma$, and the parameters λ_j are also grouped into a string structure similar to (17), $\lambda_j \sim \lambda_0 + \mathrm{i}(\gamma/2)(n+1-2j)$.

Even strings ($n = 2k$) obviously exist at arbitrary value $h^{(\alpha,n)}$. In an odd string ($n = 2k+1$), one of the rapidities, say $h(\mu)$, lies on the real axis, therefore the corresponding wavefunction $\phi(x|\mu)$ vanishes for μ lying within the gap. The only exception is $\mu = \Lambda$, where Λ is the eigenenergy of the discrete mode. In this case, one can build an odd string:

$$h_j = \tfrac{1}{2}\mathrm{i}(n+1-2j) \qquad n = 2k+1 \tag{19}$$

which is pinned to the atom and describes the many-polariton–atom bound state, in which the radiation and the medium polarization are localized in the vicinity of the atom.

We hope that the hidden integrability, which has been demonstrated here with the Dicke model in a dispersive medium, can also be found in other physical systems with non-local coupling. The results obtained also bring up the intriguing question: could one construct an integrable system with non-local coupling using an integrable system with local coupling as an auxiliary one?

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